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A POINT DISTAL TRANSFORMATION OF THE TORUS

BY

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ABSTRACT

A point-distal non-distal homeomorphism of the torus is constructed. By a similar construction, a point-distal homeomorphism of the $n+1$–dimensional torus can be constructed, with any two compact subsets of $\mathbf{R}^n$ among its fibres over some factor.

Let $K = \mathbf{R}/\mathbf{Z}$ denote the circle, and K^2 the torus. The basic proposition is the following:

PROPOSITION. *Let T be a minimal almost periodic homeomorphism of K^2 of the form*:

$$T(x, y) = (x + \alpha, y + \beta).$$

Let $(x_0, y_0) \in K^2$. Then there exists a homeomorphism $S: K^2 \to K^2$ such that $(K^2, T) <_\Phi (K^2, S)$, where Φ is of the form $\Phi(x, y) = (x, \varphi(x, y))$ and $\Phi^{-1}(x, y)$ is singleton except *when $(x, y) = T^n(x_0, y_0)$ for some n, in which case $\Phi^{-1}(x, y)$ is an interval in $\{x\} \times K$. Thus S is a minimal point-distal (non-distal) homeomorphism.*

The purpose of this construction is to provide a non-pathological example of a point-distal non-distal minimal homeomorphism. Up to now examples have tended to be constructed on zero-dimensional spaces [1].

Various generalizations will be stated at the end of the paper. As a corollary, it can be shown that any two compact subsets of $\mathbf{R}^n$ can occur among the fibres of a minimal transformation group with phase space K^{n+1} over some factor (see 4.3). This answers a question raised by Furstenberg in private correspondence.

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1. Reduction of proof of the proposition.

1.1. First we need some notation. Let $z_0=(x_0,y_0)\in K^2$, and $T\colon K^2\to K^2$ be fixed from now on, where:

$$T(x,y)=(x+\alpha,y+\beta).$$

Let $X=(K\setminus\bigcup_{i=-\infty}^{\infty}\{x_0+i\alpha\})\times K$. For $n\geqq 0$, let $z_{2n}=(x_{2n},y_{2n})=T^n(x_0,y_0)$ and

$$z_{2n+1}=(x_{2n+1},y_{2n+1})=T^{-n-1}(x_0,y_0).$$

Let $\sigma,\tau\colon \mathbf{N}\to\mathbf{N}$ be bijective maps such that $Tz_n=z_{\sigma(n)}$ and $T^{-1}z_n=z_{\tau(n)}$. So $\sigma\circ\tau=\tau\circ\sigma=$ identity on $\mathbf{N}$. Then $0<|\sigma(n)-n|\leqq 2$ and $0<|\tau(n)-n|\leqq 2$ for all n.

For $x,y\in K$, if $x=\mathbf{Z}+x_1$, $y=\mathbf{Z}+y_1$ $(x_1,y_1\in\mathbf{R})$, then let $|x-y|=\inf_{p\in\mathbf{Z}}|x_1-y_1+p|$.

If $x=(x_1\cdots x_r)\in K^r$, $y=(y_1\cdots y_r)\in K^r$, let $|x-y|=\mathrm{Max}_{1\leqq i\leqq r}|x_i-y_i|$.

If Y is any topological space and $f,g\colon Y\to K^r$, then let

$$\|f-g\|_\infty=\sup_{y\in Y}|f(y)-g(y)|.$$

For $A\subseteq K^r$, let $|A|=\sup\{|a-b|\colon a,b\in A\}$.

1.2. *First Reduction.* To prove the proposition, it will suffice to construct $\Phi\colon K^2\to K^2$ satisfying (i)–(iii) below:

(i) Φ is a continuous onto map of the form $\Phi(x,y)=(x,\varphi(x,y))$.

(ii) $\Phi^{-1}(z)$ is singleton for $z\neq z_n$ $(n\geqq 0)$ and $\Phi^{-1}(z_n)$ is an interval in $\{x_n\}\times K$.

(If Φ satisfies (i) and (ii) then the dense subset $X\subseteq K^2$ of 1.1 is Φ- and T-invariant, and Φ maps X one-to-one onto itself.)

(iii) The maps $\Phi^{-1}\circ T\circ\Phi$, $\Phi^{-1}\circ T^{-1}\circ\Phi\colon X\to X$ are uniformly continuous.

For if (i), (ii) and (iii) are satisfied, let S be the unique extension of $\Phi^{-1}\circ T\circ\Phi$ to K^2.

1.3. *Second Reduction.* To prove the proposition, it will suffice to construct a sequence $\{A_n\}_{n\geqq 0}$ of closed subsets of K^2, and a sequence $\{\Lambda_n\}_{n\geqq 0}$ of continuous functions from K^2 to K^2 such that:

(i) A_n is of the form $\{(\mathbf{Z}+x,\mathbf{Z}+y)\colon x\in[a_n,b_n],\ y\in[c_n,d_n]\}$, where $z_n=(\mathbf{Z}+(a_n+b_n)/2,\ \mathbf{Z}+(c_n+d_n)/2)$, so that z_n is the centre of A_n.

(ii) $\Lambda_n=$ identity on $K^2\setminus A_n$; $\Lambda_n(A_n)=A_n$; Λ_n is continuous and of the form $\Lambda_n(x,y)=(x,\lambda_n(x,y))$; $\Lambda_n^{-1}(x,y)$ is singleton except if $(x,y)=z_n$, and:

$$\Lambda_n^{-1}(z_n)=\{x_n\}\times\{\mathbf{Z}+y\colon y\in[(3c_n+d_n)/4,(c_n+3d_n)/4]\}.$$

(iii) $\Lambda_{\sigma(n)}^{-1}\circ T\circ\Lambda_n$ and $\Lambda_{\tau(n)}^{-1}\circ T^{-1}\circ\Lambda_n$ are uniformly continuous on $K^2\setminus\{z_n\}$.

(iv) $|A_n| < (\frac{1}{2})\,\mathrm{Max}\{|x_m - x_p|: m \neq p, m, p \leqq n+5\}$. (So Λ_n and Λ_m will commute for $|m-n| \leqq 5$.)

(v) $|A_n| < 1/2^n$.

(vi) There exists a sequence $\{a_n\}_{n=1}^{\infty}$ of integers $\geqq 2$ with $\Pi_{n=1}^{\infty}(1-(1/a_n)) > 0$, and if $x, y \in K^2$ with $|x-y| < 1/n$, then for $m > n$,

$$|\Phi_m(x) - \Phi_m(y)| \geqq \prod_{r=n+1}^{m} (1-(1/a_r))|\Phi_n(x) - \Phi_n(y)|$$

where $\Phi_n: K^2 \to K^2$ is $\Lambda_n \circ \Lambda_{n-1} \circ \cdots \circ \Lambda_0$.

(vii) Let $\Phi_{\sigma,n} = \Lambda_{\sigma(n)} \circ \cdots \circ \Lambda_{\sigma(0)}$, $\Phi_{\tau,n} = \Lambda_{\tau(n)} \circ \cdots \circ \Lambda_{\tau(0)}$.

Let $\Gamma_n, \Delta_n: X \to X$ be defined by $\Gamma_n = \Phi_{\sigma,n}^{-1}\circ T\circ\Phi_n$, $\Delta_n = \Phi_{\tau,n}^{-1}\circ T^{-1}\circ\Phi_n$. Then each Γ_n, Δ_n is uniformly continuous on X, and $\{\Gamma_n\}$, $\{\Delta_n\}$ are uniformly Cauchy on X.

PROOF. $\|\Phi_{n+1} - \Phi_n\|_\infty = \|\Lambda_{n+1}\text{-identity}\|_\infty < 1/2^{n+1}$ by (v). So Φ_n converges uniformly to Φ, say.

(iv) and (v) imply that the sequences $\{\Phi_{\sigma,n}\}$ and $\{\Phi_{\tau,n}\}$ also converge uniformly to Φ.

If the sequences Γ_n and Δ_n converge uniformly on X to Γ and Δ respectively (see (vii)), then (vii) implies that Γ and Δ are uniformly continuous on X, and it is easily seen that $\Phi\circ\Gamma = T\circ\Phi$, $\Phi\circ\Delta = T^{-1}\circ\Phi$ on X. Hence the second reduction implies the first reduction provided we can prove the following lemma:

1.4. LEMMA. 1.3 *implies* (ii) *of* 1.2.

PROOF. Suppose $\Phi(x) = \Phi(y)$, some $x \neq y$. $|x-y| \geqq 1/n$, say. Then

$$0 = |\Phi(x) - \Phi(y)| = \lim_{m\to\infty}|\Phi_m(x) - \Phi_m(y)| \geqq \left\{\prod_{r=n+1}^{\infty}(1-(1/a_r))\right\}|\Phi_n(x) - \Phi_n(y)|$$

by (vi) of 1.3.

So $\Phi_n(x) = \Phi_n(y) = z_i$, some $i \leqq n$, by (ii), (iv) of 1.3.

So $\Phi(x) = \Phi(y) = z_i$ by (iv) of 1.3.

1.5. Clearly a sequence $\{\varepsilon_n^1\}_{n=0}^{\infty}$ of positive reals can be chosen such that if $|A_n| < \varepsilon_n^1$ then (i), (iv), (v) of 1.3 are satisfied. In section 2 we shall show how to construct $\{\Lambda_n\}_{n=0}^{\infty}$, given $\{A_n\}_{n=0}^{\infty}$, such that (ii) and (iii) are satisfied. In section 3 we shall show that by choosing $|A_n|$ sufficiently small, (vi) and (vii) will be satisfied. Hence the proof of the proposition will be completed.

2. Construction of the Λ_n

2.1. We shall define a function $A \mapsto \Lambda_A$ from the set of closed "squares" in K^2 to $C(K^2, K^2)$, the set of continuous functions from K^2 to K^2 such that, if $\{A_n\}$ is a sequence of squares with z_n the centre of A_n, we shall define $\Lambda_n = \Lambda_{A_n}$, and then (ii) and (iii) of 1.3 will be satisfied.

We shall ensure that Λ_A has the following properties:

Let $A = \{(\mathbf{Z} + x, \mathbf{Z} + y)\colon (x, y) \in [a, b] \times [c, d]\}$ be denoted by $[a, b] \times [c, d]$ by abuse of notation:

(i) $\Lambda_A : K^2 \to K^2$ is continuous onto.

(ii) Λ_A = identity outside $A, \Lambda_A(A) = A$.

(iii) Λ_A is of the form $\Lambda_A(x, y) = (x, \lambda_A(x, y))$.

(iv) $\Lambda_A^{-1}(x, y)$ is singleton except for $(x, y) = (\mathbf{Z} + (a+b)/2, \mathbf{Z} + (c+d)/2) = z_A$, in which case $\Lambda_A^{-1}(x, y) = \{\mathbf{Z} + (a+b)/2\} \times [(3c+d)/4, (c+3d)/4] = I_A$.

(v) If $R\colon K^2 \to K^2$ is defined by:

$$R(z_A + \boldsymbol{x}) = z_B + \boldsymbol{x} \quad (\boldsymbol{x} \in K^2),$$

then $\Lambda_B^{-1} \circ R \circ \Lambda_A$ is uniformly continuous on its domain of definition $K^2 \setminus \{z_A\}$.

2.2. Definition. If $[-1, 1]$ denotes the closed interval *of* $\mathbf{R}$, choose a continuous function $h\colon [-1, 1]^2 \to [-1, 1]$ with the following properties, where $h_t(s) = h(t, s)$ (by definition):

(i) h_t is a homeomorphism for $t \neq 0$.

(ii) $h_1 = h_{-1}$ = identity on $[-1, 1]$. $h_t(1) = 1$, $h_t(-1) = -1$ for all t.

(iii) $h_t([-\frac{1}{2}, \frac{1}{2}]) = [-|t|/2, |t|/2]$ for all $t \neq 0$.

$$[-\tfrac{1}{2}, \tfrac{1}{2}] = h_0^{-1}(\{0\}).$$

(iv) h_0 is onto, and $h_0^{-1}(t)$ is singleton for $t \neq 0$.

(v) $h_t = h_0$ on $h_0^{-1}([-1, -t] \cup [t, 1])$.

2.3. Definition of Λ_A. Suppose $|A| = 2\delta$. Define Λ_A = identity outside A, and:

$$\Lambda_A(z_A + (t, s)) = z_A + (t, h_t \circ h_\delta^{-1}(s)) \quad \text{for all } (t, s) \in [-\delta, \delta]^2.$$

It is easily checked that Λ_A is the identity on the boundary of A, hence continuous, and that (i)–(v) of 2.2 are satisfied.

For example, to show (v) is satisfied, it suffices to show that $\Lambda_B^{-1} \circ R \circ \Lambda_A$ is uniformly continuous on $(A \cap R^{-1}(B)) \setminus \{z_A\}$. But if $z_A + \boldsymbol{x}$ is in this set, with $\boldsymbol{x} = (x_1, x_2)$,

$$\Lambda_B^{-1}\circ R\circ\Lambda_A(z_A+x)=\Lambda_B^{-1}(z_B+(x_1,h_{x_1}\circ h_\delta^{-1}(x_2))$$

$$=z_B+(x_1,h_\eta\circ h_{x_1}^{-1}\circ h_{x_1}\circ h_\delta^{-1}(x_2))=z_B+(x_1,h_\eta\circ h_\delta^{-1}(x_2))$$

where $2\eta=|B|$ and $2\delta=|A|$.

(v) follows since the function $h_\eta\circ h_\delta^{-1}$ is uniformly continuous.

3. Completion of proof of the proposition

3.1. The $\{A_n\}$ and $\{\Lambda_n\}=\{\Lambda_{A_n}\}$ (see section 2) are defined inductively. Suppose $|A_r|<\mathrm{Min}(\varepsilon_r^1,\varepsilon_r^2,\varepsilon_r^3)$ for $r\geqq 0$, where the ε_r^2 and ε_r^3 are defined below. It will be shown that the $\{A_n\}$ and $\{\Lambda_n\}$ satisfy (vi) and (vii) of 1.3, which will complete the proof of the proposition.

(i) $\varepsilon_0^2=\varepsilon_0^1$, and for $r\geqq 0$, ε_r^2 is chosen so that if $|A_r|<\varepsilon_r^2$, then: $|\Phi_{r-1}^{-1}((a_r+1)A_r)|<1/r$, where $\{a_r\}$ is a fixed sequence of integers with $a_r\geqq 2$ and $\Pi_{r=1}^{\infty}(1-(1/a_r))>0$, and $(a_r+1)A_r$ denotes the square with the same centre as, and sides (a_r+1)-times as long as, A_r.

(ii) $\varepsilon_0^3=\varepsilon_0^1$.

For $s\geqq 0$, a closed subset N_s of K^2 is chosen so that $z_i\notin N_s$ for $i\leqq s$ and $A_i\subseteq N_s$ $(s+1\leqq i\leqq s+5)$ and $T(A_{s+3})\subseteq N_s$, $T^{-1}(A_{s+3})\subseteq N_s$ (this is possible by 1.3 (iv)).

Then Φ_s^{-1} is defined on N_s. So η_s is chosen so that:

$$\text{if } x,y\in N_s \text{ and } |x-y|\leqq\eta_s,\text{ then } |\Phi_s^{-1}(x)-\Phi_s^{-1}(y)|<1/2^s.$$

Then $\varepsilon_r^3=\mathrm{Min}(\eta_{r-i}:1\leqq i\leqq 5)$.

3.2. Lemma *If $|A_n|<\varepsilon_n^2$ for all n, then* (vi) *of* 1.3 *will be satisfied.*

Proof Let $|x-y|\geqq 1/n$. Then for $m\geqq n$, $\Phi_m(x)$ and $\Phi_m(y)$ are not both in $(a_{m+1}+1)A_{m+1}$.

Case 1. If $\Phi_m(x)$ and $\Phi_m(y)$ are both not in A_{m+1} then

$$|\Phi_{m+1}(x)-\Phi_{m+1}(y)|=|\Phi_m(x)-\Phi_m(y)|.$$

Case 2. If $\Phi_m(x)\in A_{m+1}$ and $\Phi_m(y)\notin(a_{m+1}+1)A_{m+1}$, then

$$|\Phi_{m+1}(x)-\Phi_m(x)|<|A_{m+1}|,\quad\text{so}\quad|\Phi_m(x)-\Phi_m(y)|\geqq a_{m+1}|A_{m+1}|.$$

So

$$|\Phi_{m+1}(x)-\Phi_{m+1}(y)|\geqq|\Phi_m(x)-\Phi_m(y)|-|\Phi_{m+1}(x)-\Phi_m(x)|$$

$$>|\Phi_m(x)-\Phi_m(y)|-|A_{m+1}|\geqq|\Phi_m(x)-\Phi_m(y)|(1-(1/a_{m+1})).$$

So by induction, (vi) of 1.3 is satisfied.

So now we only need to show (vii) of 1.3 can be satisfied. This will be shown in 3.3–3.5.

In 3.3, 3.4, $\| \ \|_\infty$ denotes the sup norm on functions defined *on X*.

3.3. LEMMA. *If* $|A_n| < \varepsilon_n^1$, *then*:

$$\|\Gamma_n - \Gamma_{n+1}\|_\infty \leqq \operatorname*{Max}_{n-1\leqq r\leqq n+3} \|\Phi_{n-2}^{-1}\circ\Lambda_r^{-1} - \Phi_{n-2}^{-1}\|_\infty + \|\Phi_{n-2}^{-1}\circ T\circ\Lambda_{n+1} - \Phi_{n-2}^{-1}\circ T\|_\infty,$$

$$\|\Delta_n - \Delta_{n+1}\|_\infty \leqq \operatorname*{Max}_{n-1\leqq r\leqq n+3} \|\Phi_{n-2}^{-1}\circ\Lambda_r^{-1} - \Phi_{n-2}^{-1}\|_\infty + \|\Phi_{n-2}^{-1}\circ T^{-1}\circ\Lambda_{n+1} - \Phi_{n-2}^{-1}\circ T^{-1}\|_\infty.$$

PROOF. 1.3(iv) shows that on the set where Γ_n and Γ_{n+1} differ,

$$\Gamma_n = \Phi_{n-2}^{-1}\circ T\circ\Phi_{n-2}, \qquad \Gamma_{n+1} = \Phi_{n-2}^{-1}\circ\Lambda_{\sigma(n+1)}^{-1}\circ T\circ\Lambda_{n+1}\circ\Phi_{n-2}.$$

So

$$\|\Gamma_n - \Gamma_{n+1}\|_\infty \leqq \|\Phi_{n-2}^{-1}\circ\Lambda_{\sigma(n+1)}^{-1}\circ T - \Phi_{n-2}^{-1}\circ T\|_\infty + \|\Phi_{n-2}^{-1}\circ T - \Phi_{n-2}^{-1}\circ T\circ\Lambda_{n+1}\|_\infty$$
$$= \|\Phi_{n-2}^{-1}\circ\Lambda_{\sigma(n+1)}^{-1} - \Phi_{n-2}^{-1}\|_\infty + \|\Phi_{n-2}^{-1}\circ T\circ\Lambda_{n+1} - \Phi_{n-2}^{-1}\circ T\|_\infty.$$

The result for $\|\Gamma_n - \Gamma_{n+1}\|_\infty$ follows since $|\sigma(n) - n| \leqq 2$ for all n, and the result for $\|\Delta_n - \Delta_{n+1}\|_\infty$ is similar.

3.4. LEMMA. *If* $|A_n| < \operatorname{Min}(\varepsilon_n^3, \varepsilon_n^1)$ *for all n, then*:

$$\|\Gamma_n - \Gamma_{n+1}\|_\infty < 1/2^{n-3} \quad \textit{and} \quad \|\Delta_n - \Delta_{n+1}\|_\infty < 1/2^{n-3}.$$

PROOF. Let N_s, η_s be as in 3.1 (i) ($s \geqq 0$).

Using 1.3 (iv), if $n-1 \leqq r \leqq n+3$, $x \in X$ and $\Lambda_r^{-1}(x) \neq x$, then $x, \Lambda_r^{-1}(x) \in N_{n-2} \cap A_r$ and $|x - \Lambda_r^{-1}(x)| < \eta_{n-2}$.

If $T\circ\Lambda_{n+1}(x) \neq T(x)$ then $T\circ\Lambda_{n+1}(x)$, $T(x) \in N_{n-2}$ and

$$|T\circ\Lambda_{n+1}(x) - T(x)| = |\Lambda_{n+1}(x) - x| < \eta_{n-2}.$$

So $\operatorname{Max}_{n-1\leqq r\leqq n+3}\|\Phi_{n-2}^{-1}\circ\Lambda_r^{-1} - \Phi_{n-2}\|_\infty < \frac{1}{2}^{n-2}$, and $\|\Phi_{n-2}^{-1}\circ T\circ\Lambda_{n+1} - \Phi_{n-2}^{-1}\circ T\|_\infty < 1/2^{n-2}$. So by 3.3 $\|\Gamma_n - \Gamma_{n+1}\|_\infty < 1/2^{n-3}$. Similarly $\|\Delta_n - \Delta_{n+1}\|_\infty < 1/2^{n-3}$.

3.5. LEMMA. Γ_n *and* Δ_n *are uniformly continuous on X.*

PROOF. Γ_0 and Δ_0 are uniformly continuous on X by the definition of Λ_n in section 2 (see in particular 2.1 (v)) and similarly $\Lambda_{\sigma(n)}^{-1}\circ T\circ\Lambda_n$ and $\Lambda_{\tau(n)}^{-1}\circ T^{-1}\circ\Lambda_n$ are uniformly continuous on X for all $n\geqq 0$.

Assume inductively that Γ_n is uniformly continuous. $\Gamma_n=\Gamma_{n+1}$ except on $[x_{n+1}-\varepsilon_{n+1}^1, x_{n+1}+\varepsilon_{n+1}^1]\times K=A$. To show Γ_{n+1} is uniformly continuous, it suffices to show uniform continuity on A. A is invariant under all the Λ_m and is mapped to

$$[x_{\sigma(n+1)}-\varepsilon_{n+1}^1,\ x_{\sigma(n+1)}+\varepsilon_{n+1}^1]\times K=B$$

by $\Lambda_{\sigma(n+1)}^{-1}\circ T\circ\Lambda_{n+1}$, and B is invariant under all Λ_m.

Φ_n is uniformly continuous on A and $\Phi_{\sigma,n}^{-1}$ is uniformly continuous on B. $\Lambda_{\sigma(n+1)}^{-1}\circ T\circ\Lambda_{n+1}$ is uniformly continuous on A.

Since $\Gamma_{n+1}=\Phi_{\sigma,n}^{-1}\circ\Lambda_{\sigma(n+1)}^{-1}\circ T\circ\Lambda_{n+1}\circ\Phi_n$, Γ_{n+1} is uniformly continuous on A, as required.

4. Generalizations of the proposition and a corollary

4.1. (First generalization) In the proposition, the construction can be carried out with T any minimal distal homeomorphism of the form:

$$T(x,y)=(x+\alpha, y+g(x)).$$

4.2. (Second generalization — to n dimensions) Let $T: K^n\to K^n$ ($n\geqq 2$) be an arbitrary minimal almost periodic homeomorphism with $T(x_1\cdots x_n)=(x_1+\alpha_1\cdots x_n+\alpha_n)$. Let $z_1\cdots z_m\in K^n$ be such that $T^r(z_i)\neq z_j$ for any r, if $i\neq j$. Then there exists a homeomorphism $S: K^n\to K^n$ such that:

$$(K^n, T)<_{\Phi}(K^n, S),$$

where $\Phi(\boldsymbol{x})=\Phi(x_1\cdots x_n)=(x_1,\varphi_2(\boldsymbol{x})\cdots\varphi_n(\boldsymbol{x}))$, and $\Phi^{-1}(z)$ is singleton except for $z=T^n(z_i)$, in which case $\Phi^{-1}(z)$ is homeomorphic to an $(n-1)$-cube.

Moreover, for each i:

$$|\Phi^{-1}(T^n(z_i))|\to 0 \qquad \text{as } |n|\to\infty.$$

4.3. Corollary. *If T, S, Φ, K^n are as in 4.2, and $B_1,\cdots B_m$ are any compact subsets of $\mathbf{R}^{n-1}$, then there exists (X, T_1) with $(K^n, T)<_{\Phi_1}(X, T_1)<_{\Phi_2}(K^n, S)$ with $\Phi_1\circ\Phi_2=\Phi$ and with $x_i\in\Phi_1^{-1}(z_i)$ $(i=1\cdots m)$ such that $\Phi_2^{-1}(T_1^n(x_i))$ is homeomorphic to B_i $(n\in\mathbf{Z})$ and $\Phi_2^{-1}(x)$ is singleton if $x\neq T^n(x_i)$ for any n, i.*

Proof. Regard B_i as a subset of $\Phi^{-1}(z_i)$. For $z, w\in K^n$, define $z\sim w$ if and only if either $z=w$ or both $z, w\in T^n(B_i)$, for some $n\in\mathbf{Z}$, $i=1\cdots m$.

Then $z \sim w$ implies $\Phi(z) = \Phi(w)$ and $\sim$ is a closed T-invariant equivalence relation (closure follows from the fact that $|\Phi^{-1}(T^n(z_i))| \to 0$ as $|n| \to \infty$).

Now put $X = K^n / \sim$ and let T_1 be the homeomorphism of X induced by T.

NOTE. X is n-dimensional (proof omitted) but not, in general, a manifold.

4.4. (Third generalization) In 4.2, $\Phi^{-1}(T^n(z_i))$ can be taken to be homeomorphic to any compact subset B_i of $\mathbf{R}^{n-1}$ with the following properties:

Regarding B_i as a subset of $(-1,1)^{n-1}$, there exists a continuous function $h:[-1,1]^n \to [-1,1]^{n-1}$, such that if $h_t(\mathbf{x}) = h(t,\mathbf{x})$ $(t \in [-1,1]$, $\mathbf{x} \in [-1,1]^{n-1})$, then:

(i) h_t is a homeomorphism for all t.

(ii) $h_1 = h_{-1} =$ identity, $h_t =$ identity on the boundary of $[-1,1]^{n-1}$ for all t.

(iii) $h_t(B_i) \subseteq (-t,t)^{n-1}$ for all $t \neq 0$.

(iv) h_0 is onto with $h_0^{-1}(\mathbf{0}) = B_i$. $h_0^{-1}(\mathbf{x})$ is singleton if $\mathbf{x} \neq 0$.

(v) $h_t = h_0$ on $h_0^{-1}([-1,1]^{n-1} \setminus (-t,t)^{n-1})$.

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